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BIFURCATION OF PLANE INCOMPRESSIBLE ELASTIC MEMBRANES SUBJECTED TO DEAD-LOAD TRACTIONS

D. M. HAUGHTON
Department of Mathematics, University of Glasgow, Glasgow G12 8QW, U.K.

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Abstract—An incompressible elastic membrane subjected to dead-load tractions is considered. General bifurcation criteria are obtained for homogeneous shape-change bifurcations, incremental modes of bifurcation and for the emergence of shear bands. The homogeneous mode of bifurcation is the first to occur and we show that incremental bifurcation modes may exist as secondary bifurcations. We find that shear bands may occur simultaneously with the incremental modes.

1. INTRODUCTION

In this paper we consider a plane membrane of incompressible, isotropic hyperelastic material subjected to dead-load tractions. Three possible modes of bifurcation and secondary bifurcation are considered. Firstly we consider a homogeneous shape-change mode of bifurcation (square to rectangle) by applying the work of Ogden[1], who considered homogeneous bifurcation for plane-strain problems, to the special plane-stress problem under consideration. A bifurcation criterion is obtained for an arbitrary form of the strain–energy function. We then consider the special case of equi-biaxial dead loading and find a simple, explicit, bifurcation criterion. The assumption of equi-biaxial dead loading defines the post-bifurcation path. Secondly we consider a local incremental bifurcation mode by looking for non-trivial solutions to the equations of small deformations superposed on large as given in Refs [2, 3]. Here the nature of the boundary conditions applied to the edges of the membrane is irrelevant and a general bifurcation criterion is again obtained for an arbitrary form of strain–energy function. For particular materials we show that the incremental bifurcation modes may be encountered on the dead loading post-bifurcation path and hence may be regarded as secondary bifurcations. This is demonstrated explicitly in a numerical example.

Finally we consider the possibility of shear bands emerging at some state of general biaxial deformation. We find that the bifurcation criterion is identical to that for the incremental modes with a different interpretation of the parameter involved. Consequently shear bands may occur simultaneously with the incremental modes. A detailed post-bifurcation analysis would be required to investigate the subsequent deformation of the membrane but that is not considered here.

2. BASIC EQUATIONS

We consider a plane uniform elastic membrane of thickness H in the undeformed configuration. We suppose that it is in a state of homogeneous biaxial deformation with principal in-plane stretches λ_1 and λ_2. Since we assume that the material is incompressible the principal stretch perpendicular to the plane of the membrane is given by

$$\lambda_3 = (\lambda_1\lambda_2)^{-1}. \qquad (1)$$

The principal Cauchy stresses are given by

$$\sigma_{ii} = \sigma_i - p = \lambda_i \frac{\partial W}{\partial \lambda_i} - p \qquad (i = 1, 2, 3 \text{ no sum}) \qquad (2)$$

where $W(\lambda_1, \lambda_2, \lambda_3)$ is the strain–energy function and p is an arbitrary hydrostatic pressure. The membrane approximation $\sigma_{33} = 0$ enables us to write

$$\sigma_{\alpha\alpha} = \lambda_\alpha \hat{W}_\alpha \qquad (\alpha = 1, 2 \text{ no sum}) \tag{3}$$

where $\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, (\lambda_1\lambda_2)^{-1})$ and $\hat{W}_\alpha = \partial\hat{W}/\partial\lambda_\alpha$.

We shall assume that the membrane is stretched by the application of dead-load tractions $\mathbf{t}$ per unit *undeformed* area given by

$$\mathbf{t} = \mathbf{s}^T\mathbf{N} \tag{4}$$

where $\mathbf{s}$ is the nominal stress tensor and $\mathbf{N}$ is the unit outward normal to the surface in the reference configuration. We note the connection

$$\mathbf{s} = \boldsymbol{\alpha}^{-1}\boldsymbol{\sigma} \tag{5}$$

for an incompressible material, where $\boldsymbol{\alpha}^{-1}$ is the inverse of the deformation gradient.

The incremental equilibrium equations[2, 3] are then

$$\operatorname{div} \dot{\mathbf{s}}_0 = 0 \tag{6}$$

where div is the divergence operator in the current finitely deformed configuration and

$$\dot{\mathbf{s}}_0 = \mathbf{B}\boldsymbol{\eta}^T + p\boldsymbol{\eta} - \dot{p}\boldsymbol{\delta}. \tag{7}$$

Here $\boldsymbol{\eta}$ is the incremental displacement gradient in the current configuration, $\dot{p}$ is the incremental hydrostatic pressure and $\mathbf{B}$ is the fourth-order tensor of instantaneous moduli whose non-zero components are given by

$$\begin{aligned}
B_{iijj} &= \lambda_i\lambda_j \frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j},\\
B_{ijij} &= \frac{(\sigma_i - \sigma_j)\lambda_i^2}{\lambda_i^2 - \lambda_j^2}, \lambda_i \neq \lambda_j, i \neq j,\\
B_{ijij} &= \tfrac{1}{2}(B_{iiii} - B_{iijj} + \sigma_i), \lambda_i = \lambda_j, i \neq j,\\
B_{ijji} &= B_{jiij} = B_{ijij} - \sigma_i, \qquad i \neq j.
\end{aligned} \tag{8}$$

We emphasize that all incremental quantities have been arranged through the current thickness of the membrane, see Refs [2, 3] for details. For an incompressible membrane we then have

$$\eta_{33} = -\eta_{11} - \eta_{22} \tag{9}$$

where the 3-direction is normal to the membrane middle surface. We also have

$$\eta_{13} = -\eta_{31}, \qquad \eta_{23} = -\eta_{32} \tag{10}$$

and the membrane assumptions require that

$$\dot{s}_{o_{33}} = 0 \tag{11}$$

which we shall use to eliminate $\dot{p}$.

3. HOMOGENEOUS BIFURCATION

In Ref. [1] Ogden considered homogeneous bifurcation for a three-dimensional body subjected to plane strain. A number of other authors have considered related problems, see Ref. [1] for details. Here we apply the work of Ref. [1] to a plane stress membrane problem. We suppose that the membrane is subjected to dead-load tractions, eqn (4), and we consider the possibility of having incremental (averaged) displacements $\mathbf{u} = (u(x_1, x_2), v(x_1, x_2), w(x_1, x_2))$ while maintaining the dead-load tractions. It is well known that, for dead loading and homogeneous incremental boundary conditions, incremental uniqueness (where $\mathbf{u} = \mathbf{0}$ is always a solution) first breaks down and bifurcation first becomes possible when

$$\mathrm{tr}(\dot{\mathbf{s}}_0 \boldsymbol{\eta}) = 0. \tag{12}$$

Using eqns (7)–(11) we can write eqn (12) in the form

$$\lambda_1^2 \hat{W}_{11} u_1^2 + \lambda_1 \lambda_2 \hat{W}_{12} u_1 v_2 + \lambda_2^2 \hat{W}_{22} v_2^2 + B_{1212} v_1^2 + 2(B_{1212} - \sigma_{11}) v_1 u_2 + B_{2121} u_2^2 + \sigma_{11} w_1^2 + \sigma_{22} w_2^2 = 0 \tag{13}$$

where $u_1 = \partial u/\partial x_1$, etc. This is in fact the condition for the underlying deformation to become neutrally stable, correct to second order in the incremental displacement $\mathbf{u}$. Shield[4] has looked at this problem from a stability point of view and has, using a different method, obtained an equation that is equivalent to eqn (13).

The membrane assumption is that $\sigma_{11} \geqslant 0$, $\sigma_{22} \geqslant 0$ and so a necessary condition for homogeneous bifurcation in tension is $w = $ constant. Since w is the normal displacement of the middle surface we can, without loss of generality, take $w = 0$. However, it should be emphasized that the incremental incompressibility condition, eqn (9), does not simplify as the average of η_{33} (which measures the change in thickness of the membrane) is not necessarily zero. Other necessary and sufficient conditions may be deduced but we now consider the special case of equi-biaxial dead loading. In this case we have $t_1 = t_2$ which always has solution $\lambda_1 = \lambda_2$. The bifurcation criterion, eqn (13), can now be simplified to

$$(2B_{1212} - \sigma_{11})\{2(u_1^2 + v_2^2) + (u_2 + v_1)^2\} + 2\{B_{1122} + B_{3333} + p - 2B_{1133}\}(u_1 + v_2)^2 + \sigma_{11}(u_2 - v_1)^2 = 0. \tag{14}$$

For most forms of strain–energy function, and for *all* of those commonly used

$$B_{1122} + B_{3333} + p - 2B_{1133} > 0$$

for all $\lambda_1 = \lambda_2 = \lambda > 1$. Consequently non-trivial solutions first become possible when

$$2B_{1212} - \sigma_{11} = 0 \tag{15}$$

and, in this case, the displacement field will be given by

$$u_1 + v_2 = 0, \qquad u_2 - v_1 = 0$$

which may be interpreted as a global shape change (square to rectangle). Since we assume that the dead-load tractions remain equal the post-bifurcation path is given by

$$t_1 = \sigma_{11}/\lambda_1 = t_2 = \sigma_{22}/\lambda_2 \qquad (\lambda_1 \neq \lambda_2) \tag{16}$$

having used eqns (4) and (5). We note that if we assume non-trivial ($\lambda_1 \neq \lambda_2$) solutions to eqn (16) and then take the limit $\lambda_1 \to \lambda_2$ we regain eqn (15) in a straightforward way.

We now consider a particular class of strain–energy functions introduced by Ogden[5]. We have

$$W(\lambda_1, \lambda_2, \lambda_3) = \mu_r(\lambda_1^{\alpha_r} + \lambda_2^{\alpha_r} + \lambda_3^{\alpha_r} - 3)/\alpha_r \tag{17}$$

where there is implied summation over $r = 1, 2, \ldots, N$, where $N \geqslant 1$ is an unspecified parameter. Also

$$\mu_r \alpha_r > 0 \quad \text{each } r. \tag{18}$$

We note that, for this particular class of strain–energy functions eqn (15) becomes

$$\mu_r\{(\alpha_r - 1)\lambda^{\alpha_r} + \lambda^{-2\alpha_r}\} = 0 \tag{19}$$

and hence a necessary condition for the existence of homogeneous bifurcation is

$$\alpha_r < 1 \quad \text{some } r. \tag{20}$$

For single-term strain–energy functions ($N = 1$) this condition is also sufficient. In particular we note that the commonly used neo-Hookean material ($\alpha_1 = 2$) is excluded from exhibiting this mode of bifurcation while the Mooney–Rivlin material ($\alpha_1 = 2$, $\alpha_2 = -2$) undergoes a homogeneous bifurcation for some value of $\lambda \in (3^{1/6}, \infty)$ depending on the ratio μ_1/μ_2.

4. SECONDARY BIFURCATION

We now consider the possibility of secondary bifurcation occurring at some point on the post-bifurcation path given by eqn (16). The incremental equations are given by eqn (6) and, following the assumptions made in the previous section, can be written in the form

$$(B_{1111} + B_{3333} + 2p - 2B_{1133})u_{11} + (B_{3333} + 2p + B_{1122} + B_{1221} - B_{1133} - B_{2233})v_{12} + B_{2121}u_{22} = 0 \tag{21}$$

$$(B_{3333} + 2p + B_{1122} + B_{1221} - B_{1133} - B_{2233})u_{12} + (B_{2222} + B_{3333} + 2p - 2B_{2233})v_{22} + B_{1212}v_{11} = 0 \tag{22}$$

having again used $\dot{s}_{033} = 0$ to eliminate $\dot{p}$. Following the usual procedure with problems of this type we set

$$u = A \sin mx_1 \cos nx_2$$

$$v = B \cos mx_1 \sin nx_2$$

where A and B are constants. Substitution into eqns (21) and (22) leads to the following condition for the existence of non-trivial solutions

$$(B_{2121} + \gamma^2\lambda_1^2\hat{W}_{11})(\gamma^2 B_{1212} + \lambda_2^2\hat{W}_{22}) - \gamma^2(\lambda_1\lambda_2\hat{W}_{12} + B_{1212} - \sigma_{11})^2 = 0 \tag{23}$$

where $\hat{W}_i = \partial\hat{W}/\partial\lambda_i$ and $\gamma = m/n$.

Because of the non-linear nature of eqn (23) little progress can be made in simplifying it even for the class of strain–energy functions (15).

We note that the bifurcation criterion, eqn (23), does not depend on the boundary conditions applied to the edges of the membrane but merely on the current state of deformation. For completeness we consider the case $\lambda_1 = \lambda_2$, even though for dead-load tractions, the homogeneous mode of bifurcation must occur first. For $\lambda_1 = \lambda_2$ it is easy to show that eqn (23) reduces to

$$(1 + \gamma^2)^2 B_{1212}(B_{1111} + B_{3333} + 2p - 2B_{1133}) = 0.$$

For m and n not both zero the bifurcation criterion then becomes

$$B_{1111} + B_{3333} + 2p - 2B_{1133} = 0 \tag{24}$$

since the Baker–Ericksen inequalities give $B_{1212} > 0$. For the strain–energy functions (17) this becomes

$$\mu_r\{(\alpha_r - 1)\lambda^{\alpha_r} + (\alpha_r + 1)\lambda^{-2\alpha_r}\} = 0 \tag{25}$$

and a necessary condition for this is

$$|\alpha_r| < 1 \quad \text{some}\, r. \tag{26}$$

For $\lambda_1 \neq \lambda_2$ we also note that eqn (23) reduces to eqn (24) for $\gamma \to \infty$ and a similar expression with subscripts 1 and 2 interchanged for $\gamma = 0$. In either case inequality (26) is again a necessary condition for the strain–energy functions, eqn (15). This suggests that the converse of inequality (26) may be sufficient to exclude infinitesimal bifurcations in the $(\lambda_1 - \lambda_2)$ plane. While numerical results obtained are in keeping with this hypothesis we are unable to prove it.

In order to illustrate the type of behaviour that is possible we consider in Fig. 1 the single-term strain–energy function, eqn (17), with $\alpha_1 = 1/2$. Condition (20) is then satisfied and we have a homogeneous shape-change bifurcation from the $\lambda_1 = \lambda_2 = \lambda$ solution path when $\lambda = 2^{2/3}$. After this value of λ is reached the $\lambda_1 = \lambda_2$ solution path becomes unstable

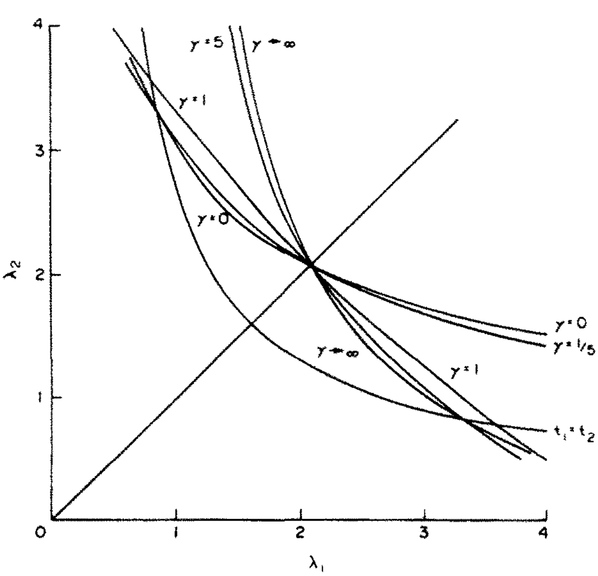

Fig. 1. A plot of the post-bifurcation path, eqn (16), and the curves of incremental bifurcation points, eqn (23), with $\gamma = 0, \frac{1}{5}, 1, 5, \infty$, for a single-term strain–energy function, eqn (17), with $\alpha_1 = \frac{1}{2}$.

and we then proceed along one branch of the post-bifurcation curve given by eqn (16), which, in this case can be written

$$\lambda_1 = \{(1 \pm [1 + 4\lambda_2^{3/2}]^{1/2})/2\lambda_2\}^2. \tag{27}$$

(An initially square membrane may elongate along the x_1 or x_2 directions giving the two possible post-bifurcation paths.) Also plotted in Fig. 1 are the curves of incremental bifurcation points satisfying eqn (23) for several values of the mode number $\gamma(= m/n)$. For this particular case we find from eqn (24) that the $\gamma = O(\infty)$ bifurcation curves are given by $\lambda_2 = 3\lambda_1^{-1/2}$ ($\lambda_1 = 3\lambda_2^{-1/2}$), respectively, and that these modes are the first to be encountered by the post homogeneous bifurcation path, eqn (27). This occurs when $\lambda_1 = (\frac{3}{4})^{2/3}$, $\lambda_2 = 6^{2/3}$ for $\gamma = 0$ and at the same values with λ_1 and λ_2 interchanged for $\gamma \to \infty$. At this point a secondary bifurcation becomes possible and a further more detailed analysis is required to determine the subsequent (local) deformation of the membrane. Finally we note that the bifurcation curves intersect on the $\lambda_1 = \lambda_2$ curve when, from eqn (24), $\lambda = 3^{2/3}$.

While Fig. 1 demonstrates the range of possible bifurcation phenomena, for more realistic forms of strain–energy function we find a more limited range of possibilities. For example, the Mooney–Rivlin strain–energy function ($\alpha_1 = 2$, $\alpha_2 = -2$) gives a shape-change bifurcation but does not exhibit incremental modes (not just on the post-bifurcation path but anywhere in the λ_1–λ_2 plane). A three-term empirical strain–energy function developed in Ref. [5] ($\alpha_1 = 1.3$, $\alpha_2 = 5$, $\alpha_3 = -2$) does not exhibit even a shape-change bifurcation (and consequently no secondary incremental modes) despite satisfying condition (20).

5. SHEAR BANDS

Finally we consider the existence of shear bands in a finitely deformed plane membrane. Shear bands have been investigated extensively in the context of plasticity and more recently in the context of finite elastic deformations. However, the possibility of their existence in elastic membranes does not appear to have been considered previously. Here we follow the work of Hill[6] and Reddy[7], where references to related work can be found.

We consider only straight line shear bands with an in-plane unit normal $\mathbf{n}$ and unit tangent ν that produce a jump in the incremental displacement field $\mathbf{u} = (u_1(x_1, x_2), u_2(x_1, x_2))$ across the line. We can then write

$$\boldsymbol{\eta} = \frac{\partial \mathbf{u}}{\partial n} \otimes \mathbf{n} + \frac{\partial \mathbf{u}}{\partial \nu} \otimes \nu$$

and

$$[\boldsymbol{\eta}] = \left[\frac{\partial \mathbf{u}}{\partial n}\right] \otimes \mathbf{n} = \mathbf{c} \otimes \mathbf{n} \tag{28}$$

where $[\cdot]$ indicates the jump in $(\cdot)$ across the shear band.

The continuity conditions for the incremental tractions across the shear band can be written

$$[\dot{\mathbf{s}}_0^T]\mathbf{n} = \mathbf{0}. \tag{29}$$

From eqns (7) and (28) the above becomes

$$c_1\{\lambda_1^2 \hat{W}_{11} n_1^2 + B_{2121} n_2^2\} + c_2 n_1 n_2 \{\lambda_1 \lambda_2 \hat{W}_{12} + B_{1212} - \sigma_{11}\} = 0 \tag{30}$$

$$c_1 n_1 n_2 \{\lambda_1 \lambda_2 \hat{W}_{12} + B_{1212} - \sigma_{11}\} + c_2\{n_1^2 B_{1212} + n_2^2 \lambda_2^2 \hat{W}_{22}\} = 0. \tag{31}$$

For non-trivial c we regain eqn (23) where we now have $\gamma = n_1/n_2$. Consequently shear modes can occur simultaneously with the incremental bifurcation modes. This contrasts with the result obtained in Ref. [7] where, for a half-space in-plane strain, it was shown that the incremental modes must occur first.

The results shown in Fig. 1 can be reinterpreted by taking $\gamma = n_1/n_2$. For example, if we follow the upper post-bifurcation curve $\lambda_2 > \lambda_1$ then the shear band $\gamma = 0$ ($n_1 = 0$, $n_2 = 1$) is the first to occur as we might expect. However, the comments at the end of Section 4 apply also to shear bands and, for most commonly used strain–energy functions, they are not likely to occur.

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